Another demonstration of the theorem by Hojman and Harleston

1983 J. Phys. A: Math. Gen. 16 707

(http://iopscience.iop.org/0305-4470/16/4/009)

J. Phys. A: Math. Gen. **16** (1983) 707–710. Printed in Great Britain

Another demonstration of the theorem by Hojman and Harleston

J Ronald Farias and L J Negri

Departamento de Física, Universidade Federal da Paraíba, 58.000, Joäo Pessoa (PB), Brazil

Received 25 August 1982

Abstract. Another demonstration of a theorem on the calculus of variations derived recently by Hojman and Harleston (using Helmholtz conditions) is presented.

1. Introduction

Recently Hojman and Harleston (1981) have demonstrated the following theorem on the calculus of variations. Consider the Lagrangian $L = L(q^i, \dot{q}^i, t)$, $i = 1, \dots, n$, such that

$$\det(\partial^2 L/\partial \dot{q}_i\, \partial \dot{q}_j) \neq 0$$

which leads to the set of equations

$$G_i \equiv (\mathrm{d}/\mathrm{d}t)(\partial L/\partial \dot{q}_i) - (\partial L/\partial q_i) = 0.$$

The Lagrangian $\bar{L} = \bar{L}(q^i, \dot{q}^i, t)$, with

$$\det(\partial^2 \bar{L}/\partial \dot{q}_i\, \partial \dot{q}_i) \neq 0,$$

is said to be subordinate (or equivalent) to L iff

$$\{G_i = 0\} \Rightarrow \{\bar{G}_j \equiv (\mathrm{d}/\mathrm{d}t)(\partial \bar{L}/\partial \dot{q}_j) - (\partial \bar{L}/\partial q_j) = 0\}.$$

Theorem. If $\bar{L}$ is subordinate to L and

$$\bar{G}_i = A_i^j(q, \dot{q}, t) G_j, \tag{1}$$

with $\det A \neq 0$, then the trace (or the trace of all integer powers) of A is a constant of the motion. (The assumption that A^{-1}, the inverse matrix to A, exists, implies that L is subordinate to $\bar{L}$, and this is assumed.)

Henneaux (1981) has derived this result in a more geometrical fashion. Lutzky (1982) also did the same using Cartan form; Gonzalez–Gascon (1982) directly from Euler's equations.

Here, we derive it using the Helmholtz conditions (Helmholtz 1887).

2. Helmholtz conditions

Helmholtz conditions are necessary and sufficient conditions for a given set of equations $G_i(q, \dot{q}, \ddot{q}, t)$, $i = 1, \ldots, n$, to be derived from Hamilton's variational principle, which are

$$\partial G_i/\partial \ddot{q}_j = \partial G_j/\partial \ddot{q}_i \tag{2a}$$

$$\frac{\partial G_i}{\partial \dot{q}_j} + \frac{\partial G_j}{\partial \dot{q}_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G_i}{\partial \ddot{q}_j} + \frac{\partial G_j}{\partial \ddot{q}_i}\right) \tag{2b}$$

$$\frac{\partial G_i}{\partial q_j} - \frac{\partial G_j}{\partial q_i} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G_i}{\partial \dot{q}_j} - \frac{\partial G_j}{\partial \dot{q}_i}\right) \qquad i, j = 1, 2, \ldots, n. \tag{2c}$$

From (2a) and (2b) it immediately follows that G_i must be linear in $\ddot{q}$. Hence, without loss of generality we can restrict ourselves to equations of the form (Engels 1975)

$$G_i(\dot{q}, \dot{q}, \ddot{q}, t) \equiv G_{ik}(q, \dot{q}, t)\ddot{q}_k + g_i(q, \dot{q}, t) = 0 \qquad i, k = 1, 2, \ldots, n \tag{3}$$

(using the summation convention). With this the Helmholtz conditions are transformed into

$$G_{ij} \equiv G_{ji} \qquad \partial G_{ik}/\partial \dot{q}_j \equiv \partial G_{jk}/\partial \dot{q}_i \tag{4a, b}$$

$$\frac{\partial g_i}{\partial \dot{q}_j} + \frac{\partial g_j}{\partial \dot{q}_i} \equiv 2\left(\frac{\partial G_{ij}}{\partial q_k}\dot{q}_k + \frac{\partial G_{ij}}{\partial t}\right) \tag{4c}$$

$$\frac{\partial G_{ik}}{\partial q_j} - \frac{\partial G_{jk}}{\partial q_i} \equiv \frac{1}{2}\left(\frac{\partial^2 g_i}{\partial \dot{q}_k\, \partial \dot{q}_j} - \frac{\partial^2 g_j}{\partial \dot{q}_k\, \partial \dot{q}_i}\right) \tag{4d}$$

$$\frac{\partial g_i}{\partial q_j} - \frac{\partial g_j}{\partial q_i} \equiv \frac{1}{2}\left(\frac{\partial^2 g_i}{\partial q_k\, \partial \dot{q}_j} - \frac{\partial^2 g_j}{\partial q_k\, \partial \dot{q}_i}\right)\dot{q}_k + \frac{\partial^2 g_i}{\partial t\, \partial \dot{q}_j} - \frac{\partial^2 g_j}{\partial t\, \partial \dot{q}_i} \qquad i, j, k = 1, 2, \ldots, n. \tag{4e}$$

The existence of $\bar{L}$ also will be assured by imposing Helmholtz conditions on

$$\bar{G}_i \equiv A_i^k G_{km}\ddot{q}_m + A_i^k g_k \equiv 0$$

which take

$$A_i^k G_{kj} = A_j^k G_{ki} \tag{5a}$$

$$\partial(A_i^k G_{km})/\partial \dot{q}_j = \partial(A_j^k G_{km})/\partial \dot{q}_i \tag{5b}$$

$$\frac{\partial(A_i^k g_k)}{\partial \dot{q}_j} + \frac{\partial(A_j^k g_k)}{\partial \dot{q}_i} = 2\left(\frac{\partial(A_i^k G_{kj})}{\partial q_m}\dot{q}_m + \frac{\partial(A_j^k G_{kj})}{\partial t}\right) \tag{5c}$$

$$\frac{\partial(A_i^m G_{mk})}{\partial q_j} - \frac{\partial(A_j^m G_{mk})}{\partial q_i} = \frac{1}{2}\left(\frac{\partial^2(A_i^m g_m)}{\dot{q}_k \partial \dot{q}_j} - \frac{\partial^2(A_j^m g_m)}{\partial \dot{q}_k \partial \dot{q}_i}\right) \tag{5d}$$

$$\frac{\partial(A_i^m g_m)}{\partial q_j} - \frac{\partial(A_j^m g_m)}{\partial q_i} = \frac{1}{2}\left[\left(\frac{\partial^2(A_i^m g_m)}{\partial q_n \partial \dot{q}_j} - \frac{\partial^2(A_j^m g_m)}{\partial q_n \partial \dot{q}_i}\right)\ddot{q}_n + \left(\frac{\partial^2(A_i^m g_m)}{\partial t \partial \dot{q}_j} - \frac{\partial^2(A_j^m g_m)}{\partial t \partial \dot{q}_i}\right)\right]. \tag{5e}$$

3. Demonstration of the theorem

Equation (5c) can be put in the form

$$\left(\frac{\partial A_i^m}{\partial \dot{q}_j}+\frac{\partial A_j^m}{\partial \dot{q}_i}\right)g_m+A_i^m\frac{\partial g_m}{\partial \dot{q}_j}+A_j^m\frac{\partial g_m}{\partial \dot{q}_i}-2\frac{\mathrm{d}}{\mathrm{d}t}(A_i^m G_{mj})+2\left(\frac{\partial}{\partial \dot{q}_n}(A_i^m G_{mj})\right)\ddot{q}_n=0. \tag{6}$$

On the other hand, using (4a), (4b), (5a) and (5b), the last term of (6) can be written as

$$2\left(\frac{\partial}{\partial \dot{q}_n}(A_i^m G_{mj})\right)\ddot{q}_n=\left(\frac{\partial A_i^m}{\partial \dot{q}_j}+\frac{\partial A_j^m}{\partial \dot{q}_i}\right)G_{mn}\ddot{q}_n+A_i^m\frac{\partial G_{mn}}{\partial \dot{q}_j}\ddot{q}_n+A_j^m\frac{\partial G_{mn}}{\partial \dot{q}_i}\ddot{q}_n. \tag{7}$$

Using (3) in (7), we get

$$2\left(\frac{\partial}{\partial \dot{q}_n}(A_i^m G_{mj})\right)\ddot{q}_n$$

$$=-\left(\frac{\partial A_i^m}{\partial \dot{q}_j}+\frac{A_j^m}{\dot{q}_i}\right)g_m-A_i^m\frac{\partial g_m}{\partial \dot{q}_j}-A_j^m\frac{\partial g_m}{\partial \dot{q}_i}-A_i^m G_{mn}\frac{\partial \ddot{q}_n}{\partial \dot{q}_j}-A_j^m G_{mn}\frac{\partial \ddot{q}_n}{\partial \dot{q}_i}.$$

Using this in (6) results in

$$2(\mathrm{d}/\mathrm{d}t)(A_i^m)G_{mj}+2A_i^m(\mathrm{d}/\mathrm{d}t)(G_{mj})+A_i^m G_{mn}(\partial \ddot{q}_n/\partial \dot{q}_j)+A_j^m G_{mn}(\partial \ddot{q}_n/\partial \dot{q}_i)=0. \tag{8}$$

Now, from (4c), using (3), we get

$$2(\mathrm{d}/\mathrm{d}t)(G_{ij})=-G_{in}(\partial \ddot{q}_n/\partial \dot{q}_j)-G_{jn}(\partial \ddot{q}_n/\partial \dot{q}_i)$$

which on substitution into (8) and using (4a) and (5a) yields the following result

$$2(\mathrm{d}/\mathrm{d}t)(A_i^m)G_{mj}-A_i^m(\partial \ddot{q}_n/\partial \dot{q}_m)G_{nj}+(\partial \ddot{q}_n/\partial \dot{q}_i)A_n^m G_{mj}=0. \tag{9}$$

Multiplying (9) by G_{jk}^{-1}, elements of the inverse matrix to G, effecting the sum on j and utilising the relation

$$G_{ij}G_{jk}^{-1}=\delta_{ik},$$

we get

$$2(\mathrm{d}/\mathrm{d}t)(A_i^k)-A_i^m(\partial \ddot{q}_k/\partial \dot{q}_m)+(\partial \ddot{q}_n/\partial \dot{q}_i)A_n^k=0. \tag{10}$$

It is easy to see from (10) that

$$2(\mathrm{d}/\mathrm{d}t)(A_k^k)=0,$$

i.e., the trace of A is a constant of motion. To prove the same result for the trace of all integer powers of A, from the above result, is trivial.

References

Engels E 1975 *Nuovo Cimento* B **26** 481
Gonzalez-Gascon F 1982 *Phys. Lett.* **87**A 385
Helmholtz H 1887 *J. reine Angew. Math.* **100** 137
Henneaux M 1981 *Hadronic J.* **4** 2137
Hojman S and Harleston H 1981 *J. Math. Phys.* **22** 1414
Lutzky M 1982 *J. Phys. A: Math. Gen.* **15** L87–91